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R. E. Barlow
University of California, Berkeley

and

A. W. Marshall
Mathematics Research Laboratory

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Summary

Sharp upper and lower bounds are derived for hazard rates and densities of distributions with monotone hazard rate. These bounds are related to Chebyshev inequalities in that they are obtained under the condition that certain moments are known. Similar bounds are also obtained when the density is a Pólya frequency function of order two.

1. <u>Introduction</u>. There is, of course, a large body of literature devoted to the subject of Chebyshev-type inequalities, which provide bounds for the probability of various events in terms of moments. In spite of this, there seems to be essentially no inequality known which gives a bound for the density in terms of moments. One reason for this is that the moments can in general be possessed by a discrete distribution. Furthermore, densities are not unique when they do exist, but may be arbitrarily defined on a set of measure zero to violate any bound.

There are, however, a number of Chebyshev inequalities known which apply to distributions F subject to restrictions which may force a density to exist, at least over part of the support of F. Furthermore, the restrictions may suggest a natural version of the density. As an example, we cite the result of Gauss [7] which requires $1 - F(x)$ to be convex in $x \geq 0$.

In Sections 2 and 4 of this paper, we obtain bounds on both the density and hazard rate under the assumption that the hazard rate is monotone. A distribution F is said to have increasing (decreasing) hazard rate denoted IHR(DHR) if $\log[1 - F(x)]$ is concave on the support of F (convex on $[0,\infty)$). In [11], it is shown that if F is IHR(DHR), then F is absolutely continuous except possibly for a jump at the right-hand (left-hand) endpoint of its interval of support. Hence F has a density, say f, with the possible addition of one saltus. The ratio $q(x) = f(x)/[1 - F(x)]$ is called the <u>hazard rate</u> of F, and if F is

IHR(DHR), there is a version of f for which q is increasing (decreasing). If F is IHR, such a version of f satisfies $f(t-) \leq f(t) \leq f(t+)$ since otherwise q would not be increasing. Therefore, we seek upper bounds on $f(t+)$ and lower bounds on $f(t-)$, and always refer to a version of the density for which q is monotone. When F is DHR, the same comments hold with "+" and "-" interchanged.

In Section 3, we obtain bounds on $q(t)$ and $f(t)$ assuming that f is a Pólya frequency density of order 2 (PF_2). Briefly, f is PF_2 if $\log f(x)$ is concave on the support of F, an interval. This condition implies that F is IHR, but not conversely. It may be regarded as a smoothness condition, and guarantees that the density is essentially unique, unimodal, and continuous within its interval of support. Properties of PF_2 densities are discussed in [9] and [12]. Many life densities such as the normal and certain of the gamma and Weibull family are PF_2.

The hazard rate has many aliases and as many uses. In reliability theory [3], $q(x)$ is also called the "failure rate", and is a measure of the quality of a device or structure at age x. In the theory of the strength of materials, it is called the "risk function", and is a function of the stress applied to a material, rather than a function of time (see, e.g. [13]). In the theory of fatigue failures, the extreme value distributions have been widely employed by Gumbel [8] and by Freudenthal and Gumbel [6]. The distributions called Type I and Type III by Gumbel have increasing hazard rate. In medical and actuarial work, the hazard rate is called the "force of mortality" or the "age specific failure rate".

In congestion theory (i.e., queueing theory, traffic theory, telephone trunking theory) the renewal density $m(t)$ (renewal rate) is of interest. For IHR distributions,

$$f(0) \leq m(t) \leq q(t)$$

(c.f. [2, p384]). Bounds on the hazard rate provided in this paper therefore also provide bounds on the renewal density.

Throughout this paper, we assume $F(0-) = 0$ and write μ_r for $\int_0^\infty x^r dF(x)$. Frequently, we use the easily verified relation

$$1 - F(x) = \exp[-\int_0^x q(z)dz].$$

Many of our arguments, particularly in Sections 3 and 4, utilize the fact that IHR distributions intertwine members of certain families of extremal distributions in rather specific ways. A fact often used is that if F and G have k moments in common, they must cross at least k times. If G is exponential in some interval, say (α,β), then because $1 - F$ is log concave and $1 - G$ is log linear on (α,β), F and G can cross at most twice in (α,β). If there are two such crossings, the first crossing of $1 - G$ by $1 - F$ must be from below, and the second must be from above.

2. <u>Bounds on densities and hazard rates in terms of a single expectation</u>.

We consider first a slight generalization of the condition that F is IHR, namely that $a(x) \equiv \theta(x)q(x)$ is increasing in $x \geq 0$, where $\theta(x) \geq 0$ for $x \geq 0$ and q is the hazard rate of F.

Several proofs of this section depend upon the fact that if $F(x) \underset{(\geq)}{\leq} G(x)$ for all x, and $\zeta(x)$ is increasing in $x \geq 0$, then

$$(2.1) \qquad \int_0^\infty \zeta(x)dF(x) \underset{(\leq)}{\geq} \int_0^\infty \zeta(x)dG(x).$$

The inequality is reversed for $\zeta(x)$ decreasing in $x \geq 0$.

We begin by defining two distributions which play the role of G in the applications of (2.1) below. Fix $t > 0$, suppose that $\theta(x) > 0$ for $x \geq t$, and let

$$1 - G_a(x) = \begin{cases} 1, & x \leq t \\ \exp\{-a \int_t^x dz/\theta(z)\}, & x > t . \end{cases}$$

In case $\theta(x) > 0$ for all $x \leq t$, let

$$1 - H_a(x) = \begin{cases} \exp\{-a \int_0^x dz/\theta(z)\}, & 0 \leq x < t \\ 0, & x > t. \end{cases}$$

If $a(x)$ is increasing in $x \geq 0$, then

$$a(x) \leq \begin{cases} a(t), & x \leq t \\ \infty, & x > t \end{cases}, \quad \text{and} \quad a(x) \geq \begin{cases} 0, & x < t \\ a(t) & x \geq t, \end{cases}$$

so that

$$q(x) \leq \begin{cases} a(t)/\theta(x), & x \leq t \\ \infty, & x > t \end{cases}, \quad \text{and} \quad q(x) \geq \begin{cases} 0, & x < t \\ a(t)/\theta(x), & x \geq t. \end{cases}$$

Hence

$$Q(x) \equiv \int_0^x q(z)dz \le \begin{cases} \int_0^x a(t)dz/\theta(z), & x \le t \\ \infty, & x > t , \end{cases}$$

and

$$Q(x) \ge \begin{cases} 0, & x < t \\ a(t) \int_t^x dz/\theta(z), & x \ge t, \end{cases}$$

or

$$1 - H_{a(t)}(x) \le 1 - F(x) \le 1 - G_{a(t)}(x). \tag{2.2}$$

Theorem 2.1. If $a(x)$ is increasing in $x \ge 0$, $\theta(x) > 0$ for $x \ge t$, and if ζ is a strictly monotone function on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu < \infty$, then there exists a unique solution a_1 of $\nu = \int_0^\infty \zeta(x)dG_{a_1}(x) \equiv \varphi_1(a_1)$ whenever $t < \zeta^{-1}(\nu)$. Furthermore,

$$f(t+) \le q(t+) \le \begin{cases} a_1/\theta(t), & t < \zeta^{-1}(\nu) \\ \infty, & t \ge \zeta^{-1}(\nu). \end{cases} \tag{2.3}$$

Proof. The second inequality is trivial and we need only consider the first. Assume that $\zeta(x)$ is increasing in x, so that by (2.1) and (2.2),

$$\nu = \int_0^\infty \zeta(x)dF(x) \le \int_0^\infty \zeta(x)dG_{a(t)}(x) \equiv \varphi_1(a(t)).$$

Clearly $\varphi_1(a)$ is strictly decreasing and continuous in a, $\lim_{a\to 0} \varphi_1(a) = \lim_{x\to\infty} \zeta(x) > \nu$, $\lim_{a\to\infty} \varphi_1(a) = \zeta(t)$. Thus, if $\nu > \zeta(t)$, there exists a unique solution a_1 of $\varphi_1(a_1) = \nu$; furthermore, $a_1 \ge a(t)$ yields (2.3).

The proof for decreasing ζ is analogous. ||

In case $\zeta(x) = x$ so that $\nu = \mu_1$, (2.3) reduces with $\theta(x) \equiv 1$ to

$$(2.4) \qquad f(t+) \leq q(t+) \leq \begin{cases} 1/(\mu_1 - t), & t < \mu_1 \\ \infty, & t \geq \mu_1, \end{cases}$$

and if $\zeta(x) = x^2$ so that $\nu = \mu_2$, (2.3) with $\theta(x) \equiv 1$ becomes

$$(2.5) \qquad f(t+) \leq q(t+) \leq \begin{cases} \dfrac{t + \sqrt{2\mu_2 - t^2}}{\mu_2 - t^2}, & t < \mu_2^{1/2} \\ \infty, & t \geq \mu_2^{1/2}. \end{cases}$$

Further explicit results are given in Theorem 2.3.

Remark. Theorem 2.1 requires that $\theta(x) > 0$ for $x \geq t$. A special case of interest is $\theta(x) = 0$, $x < x_0 < t$, and $\theta(x) = 1$, $x \geq x_0$, so that the hypothesis that a is increasing becomes the hypothesis that q is increasing in $x \geq x_0$. Thus Theorem 2.1 can be applied to the case that q is actually initially decreasing.

Before discussing the sharpness of (2.3), we prove

Theorem 2.2. If $a(x)$ is increasing in $x \geq 0$, $\theta(x) > 0$ for $x \leq t$, and if ζ is a strictly monotone function on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu < \infty$, then there exists a unique solution a_2 of $\nu = \int_0^\infty \zeta(x)dH_{a_2}(x) \equiv \varphi_2(a_2)$ whenever $t > \zeta^{-1}(\nu)$. Furthermore

$$(2.6) \qquad q(t-) \geq \begin{cases} a_2/\theta(t), & t > \zeta^{-1}(\nu) \\ 0, & t \leq \zeta^{-1}(\nu), \end{cases}$$

and

(2.7) $\quad f(t-) \geq 0.$

Proof. We prove (2.6) only for $\zeta(x)$ increasing, in which case it follows from (2.1) and (2.2) that

$$\nu = \int_0^\infty \zeta(x)dF(x) \geq \int_0^\infty \zeta(x)dH_{a(t)}(x) = \varphi_2(a(t)).$$

Clearly $\varphi_2(a)$ is strictly decreasing and continuous in a, $\lim_{a\to 0} \varphi_2(a) = \zeta(t)$, $\lim_{a\to\infty} \varphi_2(a) = \zeta(0) < \nu$. Thus, if $\zeta(t) > \nu$, there exists a unique solution a_2 of $\varphi_2(a) = \nu$; furthermore, $a_2 \leq a(t)$ and this yields (2.6). ||

Remark. Theorem 2.2 requires that $\theta(x) > 0$, $x \leq t$.

In case $\theta(x) \equiv 1$ and $\zeta(x) = x$ so that $\nu = \mu_1$, a_2 can be obtained from Table I of [1], where $e^{-a_2 t/\mu_1}$ is tabulated.

Although H_{a_2} does not have a density at t, we can still define two versions of its hazard rate by

$$q_H^-(x) = \begin{cases} \lim_{\Delta\uparrow 0}[H_{a_2}(x+\Delta)-H_{a_2}(x)]/\Delta[1-H_{a_2}(x)], & x \leq t \\ \infty, & x > t, \end{cases}$$

and

$$q_H^+(x) = \begin{cases} \lim_{\Delta\downarrow 0}[H_{a_2}(x+\Delta)-H_{a_2}(x)]/\Delta[1-H_{a_2}(x)], & x < t \\ \infty, & x \geq t\ . \end{cases}$$

Similarly, let q_G^- (q_G^+) be the left (right) continuous hazard rate of G_{a_1}.

<u>Theorem 2.1'</u>. If $t < \zeta^{-1}(\nu)$, equality in (2.3) is attained uniquely by the hazard rate q_G^+. If $t \geq \zeta^{-1}(\nu)$, equality in the right side of (2.3) is attained by the hazard rate q_H^+; the bound on $f(t)$ can be approached arbitrarily closely by distributions of the form

$$(2.8) \qquad 1 - G(x) = \begin{cases} e^{-b_1 x}, & 0 \leq x \leq t \\ e^{-b_1 t - b_2(x-t)}, & t \geq x, \end{cases}$$

where b_1 satisfies $\int_0^\infty \zeta(x)dG(x) = \nu$ and $b_2 \to \infty$.

<u>Theorem 2.2'</u>. If $t \leq \zeta^{-1}(\nu)$, equality in (2.6) and (2.7) is attained by the hazard rate q_G^-. If $t > \zeta^{-1}(\nu)$, equality in (2.6) is attained uniquely by the hazard rate q_H^-. For $t > \zeta^{-1}(\nu)$, equality in (2.7) is approximated by the distributions given in (2.8) with $b_2 \to \infty$.

We omit the proofs of Theorems 2.1' and 2.2'. It is straightforward to verify that the given distributions attain equality. Uniqueness follows from an examination of the proofs of Theorems 2.1 and 2.2.

<u>Remark</u>. Since the density of G_{a_1} is PF_2 (indeed PF_∞), the above bounds which are attained by q_G cannot be improved with this additional assumption. However, the non-trivial lower bound of (2.6) is attained by q_H, and since H does not have a PF_2 density, (2.6) can be improved when f is PF_2. Also in this case, f has a non-trivial lower bound at $\zeta^{-1}(\nu)$ (See Section 4).

Although the bounds of Theorems 2.1 and 2.2 are sharp, they are not explicit; the following theorem gives an explicit result that is sharp in only very special cases.

Theorem 2.3. If F is IHR, and $\mu_r = \int_0^\infty x^r dF(x)$, where $r \geq 1$, then

$$(2.9) \qquad f(t+) \leq q(t+) \leq \frac{[\Gamma(r+1)]^{1/r}}{\mu_r^{1/r} - t}, \qquad 0 \leq t < \mu_r^{1/r}.$$

The inequalities are sharp for $r = 1$, and in case $t = 0$, for $r \geq 1$.

Proof. Since $q(x)$ is increasing in x,

$$q(t) \leq \frac{1}{\mu_r^{1/r} - t} [Q(\mu_r^{1/r}-) - Q(t)]$$

where $Q(x) = \int_0^x q(z)dz$. The right-hand inequality follows from this, and the bound $1 - F(\mu_r^{1/r}) \geq \exp\{-[\Gamma(r+1)]^{1/r}\}$ ([4], Theorem 3.8). In case $r = 1$, (2.9) reduces to (2.4), which is sharp. Equality is attained in (2.9) with $t = 0$ by the exponential distribution with r^{th} moment μ_r. ||

Remark. The inequality

$$f(0) \leq \frac{\lambda_{i-1}\lambda_{j-1}}{\lambda_{i+j-1}}, \qquad i,j = 1,2,\ldots$$

was given in [2, p383] where $\lambda_r = \mu_r/\Gamma(r+1)$. Bounds on $f'(0)$ assuming f is PF_2 and $f(0) = 0$ are given in [9, p1030]. Additional bounds can be given on derivatives of f at $t = 0$ assuming higher order total positivity conditions [10].

In case $a(x) = \theta(x)q(x)$ is decreasing, the results possible are more limited than in the increasing case. We obtain only upper bounds for $q(t)$ and $f(t)$ under restricted conditions, and give some examples to show the impossibility of certain other non-trivial results.

Let $\theta(x) > 0$ for all $x \geq 0$, and let

$$1 - K_a(x) = \begin{cases} \exp\{-a \int_0^x dz/\theta(z)\}, & x \leq t \\ \exp\{-a \int_0^t dz/\theta(z)\}, & x > t. \end{cases}$$

If $a(x) = \theta(x)q(x)$ is decreasing in $x \geq 0$, $a(x) \geq a(t)$ for $x \leq t$, and $a(x) \geq 0$ for $x > t$. Hence $Q(x) = \int_0^x q(z)dz \geq a(t) \int_0^x dz/\theta(z)$ for $x > t$, so that

$$(2.10) \qquad 1 - F(x) \leq 1 - K_{a(t)}(x).$$

<u>Theorem 2.4</u>. If $a(x)$ is decreasing in $x \geq 0$ and ζ is a strictly decreasing function on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu < \infty$, then there exists a unique solution a_3 of $\nu = \int_0^\infty \zeta(x)dK_{a_3}(x) \equiv \varphi_3(a_3)$. Furthermore

$$(2.11) \qquad q(t+) \leq a_3/\theta(t).$$

<u>Proof</u>. By (2.1), (2.10) and the fact that ζ is decreasing, $\nu \geq \varphi_3(a(t))$. Clearly $\varphi_3(a)$ is strictly increasing and continuous in a, $\lim_{a\to\infty} \varphi_3(a) = \zeta(0) > \nu$, $\lim_{a\to 0} \varphi_3(a) = \lim_{x\to\infty} \zeta(x) < \nu$. Hence a_3 exists uniquely and since $\nu \geq \varphi_3(a(t))$, $a(t) \leq a_3$. ||

<u>Theorem 2.5</u>. If $a(x)$ is decreasing in $x \geq 0$ and if ζ is a strictly increasing function on $[0,\infty)$ such that $\lim_{x\to\infty} \zeta(x) = M < \infty$ and $\int_0^\infty \zeta(x)dF(x) = \nu < \infty$, then there exists a unique solution a_4 of $\nu = \int_0^t \zeta(x)dK_{a_4}(x) + MK_{a_4}(t) \equiv \varphi_4(a_4)$; furthermore,

$$(2.12) \qquad q(t) \leq a_4/\theta(t).$$

<u>Proof</u>. By (2.1), (2.10) and the fact that ζ is increasing, $\nu \leq \varphi_4(a(t))$. Clearly $\varphi_4(a)$ is decreasing and continuous in a, $\lim_{a\to\infty} \varphi_4(a) = \zeta(0)$,

$\lim_{a \to 0} \varphi_4(a) = M$. Hence a_4 exists uniquely and since $\nu \leq \varphi_4(a(t))$, (2.12) follows. ||

Equality is attained in (2.11) uniquely by the hazard rate of the (improper) distribution K_{a_3}, and in (2.12) by K_{a_4}, so that (2.11) and (2.12) are sharp.

The following theorem provides upper bounds for the density in case F is DHR. The proof given is quite unlike the preceding proofs, but is similar to the kind of argument used in [5].

Theorem 2.6. If F is DHR and ζ is a monotone function on $[0,\infty)$ such that $\int_{0-}^{\infty} \zeta(x)dF(x) = \nu < \infty$, then

$$(2.13) \qquad f(t-) \leq \max\left[\sup_{0<\alpha\leq} a\alpha e^{-at},\ \sup_{b\geq a^*} be^{-bt}\right],$$

where for each α, $a \equiv a(\alpha)$ satisfies

$$(2.14) \qquad a\alpha \int_0^{\infty} \zeta(x)e^{-ax}dx + (1-\alpha)\zeta(0) = \nu,$$

and $a^* = a(1)$ is determined by $a^* \int_0^{\infty} \zeta(x)e^{-a^*x}dx = \nu$.

Proof. Let

$$1 - L_\alpha(x) = \begin{cases} \alpha e^{-ax}, & x > 0 \\ 1, & x \leq 0, \end{cases}$$

where for each α, a is determined by (2.14), and suppose that for all α, $F \neq L_\alpha$. Since L_1 and F satisfy $\int \zeta dL_1 = \int \zeta dF = \nu$, $1 - L_1$ and $1 - F$ must cross at least once (otherwise, by (2.1), we obtain a contradiction). Since $1 - F$ is log convex and $1 - L_1$ is log linear, $1 - L_1$ can cross $1 - F$ only once, and this crossing must be from above. Denote by u_α

the point where $1 - L_\alpha$ crosses $1 - F$ from above if such a crossing exists, and otherwise let $u_\alpha = \infty$. By the log convexity of $1 - F$ and log linearity of $1 - L_\alpha$, it can be shown that u_α is continuous in α. Furthermore, since $\lim_{\alpha \to 0} 1 - L_\alpha(x) = 0$ for all $x > 0$, we conclude that $\lim_{\alpha \to 0} u_\alpha = \infty$.

Now let $1 - M_b(x) = e^{-bx}$, $x \geq 0$, so that $M_{a*} = L_1$ and $1 - M_{a*}$ crosses $1 - F$ from above at u_1. Let v_b be the crossing of $1 - F$ by $1 - M_b$ from above if such a crossing exists; otherwise, let $v_b = 0$. Then since $\lim_{b \to \infty} 1 - M_b(x) = 0$ for all $x > 0$, $\lim_{b \to \infty} v_b = 0$, and again it can be shown that v_b is continuous in b.

Thus we have shown that for each point $t \geq v_1$, there exists α such that $1 - L_\alpha$ crosses $1 - F$ from above at t. But this means that at t, the density of F is less than the density of L_α, or $f(t) \leq a\alpha e^{-at}$ where a is determined by (2.14). Similarly, for each point $t \leq v_1$, there exists b such that $1 - M_b$ crosses $1 - F$ from above at t, and $f(t) \leq be^{-bt}$. ||

<u>Theorem 2.6'</u>. If $\zeta(x)$ is increasing in x, equality can be attained in (2.13).

<u>Proof</u>. If the bound is attained by $\sup_{0<\alpha<1} a\alpha e^{-at} = a\alpha_0 e^{-at}$, then equality is attained by the density of L_{α_0}. If the bound is attained by $\sup_{b \geq a*} be^{-bt} = b_0 e^{-b_0 t}$, then equality is attained by the distribution

$$1 - M(x) = \begin{cases} e^{-b_0 x}, & 0 \leq x \leq t + \epsilon \\ e^{-b_0(t+\epsilon)-c(x-t-\epsilon)}, & x \geq t + \epsilon \end{cases}$$

where c is determined by the moment condition $\int \zeta(x)dM(x) = \nu$. Such a distribution is DHR, (i.e., $c < b_0$), since $\int_0^\infty \zeta(x)b_0e^{-b_0x}dx < \nu$. ||

Corollary 2.7. If F is DHR and $\int_0^\infty x^r dF(x) = \mu_r < \infty$, then

$$(2.15)\qquad f(t-) \le \begin{cases} (te)^{-1}, & t \le \lambda_r^{1/r} \\ \lambda_r^{-1/r} e^{-t/\lambda_r^{1/r}}, & \lambda_r^{1/r} \le t \le (r+1)\lambda_r^{1/r} \\ \lambda_r\left(\frac{r+1}{t}\right)^{r+1} e^{-(r+1)}, & t \ge (r+1)\lambda_r^{1/r}, \end{cases}$$

where $\lambda_r = \mu_r/\Gamma(r+1)$.

Proof. This result is a direct application of Theorem 2.6. ||

Theorem 2.8. If F is DHR, $\mu_r = \int_0^\infty x^r dF(x)$, and $F(0+) = 0$, then

$$(2.16)\qquad f(0+) = r(0+) \ge \lambda_r^{-1/r}.$$

Proof. Since $Q(x) = -\log(1 - F(x))$ is concave, $Q(x)/x$ is decreasing in x, and $q(0) = \lim_{x\to 0} Q(x)/x \ge Q(\mu_r^{1/r})/\mu_r^{1/r}$. But $1 - F(\mu_r^{1/r}) \le \le e^{-[\Gamma(r+1)]^{1/r}}$ [4], and the result follows. ||

Note that equality is attained in (2.16) by the exponential distribution.

In order to construct examples showing the impossibility of certain other non-trivial bounds, we consider

$$1 - N(x) = \begin{cases} e^{-bx}, & x \le z \\ e^{-[bz+c(x-z)]}, & x \ge z \end{cases}$$

where b and c jointly satisfy the moment condition

$$(2.17)\qquad v = \int_0^\infty \zeta(x)dN(x) = \alpha(z\ ,b) + \beta(z,b)\gamma(z,c),$$

$$\alpha(z,b) = \int_0^z \zeta(x)be^{-bx}dx,\ \beta(z,b) = e^{-bz},\ \gamma(z,c) = c\int_z^\infty \zeta(x)e^{-c(x-z)}dx.$$

Note that $\lim_{b\to\infty} \alpha(z,b) = \zeta(0)$, $\lim_{b\to 0} \alpha(z,b) = 0$, $\lim_{c\to\infty} \gamma(z,c) = \zeta(z)$ and $\lim_{c\to 0} \gamma(z,c) = \lim_{x\to\infty} \zeta(x)$. Clearly N has hazard rate

$$q_N(x) = \begin{cases} b, & x < z \\ c, & x > z. \end{cases}$$

In the following, q is decreasing and $\int_{0-}^\infty \zeta(x)dF(x) = v$.

(i). <u>If ζ is decreasing and $t > 0$, then $q(t) \geq f(t) \geq 0$ is sharp</u>: Choose ϵ, $v > \epsilon > \lim_{x\to\infty} \zeta(x)$, and c sufficiently small tha+ $\alpha(z,0) + \beta(z,0)\gamma(z,c) < \epsilon$. But $\lim_{b\to\infty}[\alpha(z,b) + \beta(z,b)\gamma(z,c)] = \zeta(0) > v$; hence by continuity, for sufficiently small c, there exists $b > c$ satisfying (2.17). With $z < t$, this shows the impossibility of non-trivial lower bounds for $q(t)$ and $f(t)$.

(ii). <u>If ζ is increasing and $\lim_{x\to\infty} \zeta(x) = M < \infty$, then $q(t) \geq 0$ is sharp</u>: Observe that $\gamma(z,c)$ increases monotoncially to M as c decreases to zero, so that $\alpha(z,b) + \beta(z,b)\gamma(z,c) < \alpha(z,b) + \beta(z,b)M$. Since $\beta(z,b)$ decreases monotonically to zero as b decreases to zero, and since $\alpha(z,b)$ is bounded and $\lim_{b\to\infty} \alpha(z,b) = 0$, we conclude that (2.17) has a solution in b for fixed c sufficiently small. With $z < t$, this shows the desired result.

(iii). <u>If ζ is increasing and $t > 0$, then $q(t) < \infty$ is sharp</u>: For fixed b sufficiently large, $\zeta(0) - \alpha(z,b)$ and $\beta(z,b)$ are arbitrarily small. Since $\zeta(0) < v$, and since $\gamma(z,c)$ is monotone in c, there exists a unique solution c of (2.17). Furthermore, $\lim_{b\to\infty} c = 0$. With $z > t$, the result follows.

With the exception of Theorem 2.6, the results of this section have been obtained by essentially the same method. While this method is straightforward, it does not always seem to be adaptable to more complex problems. In Sections 3 and 4, we therefore utilize a third method, more closely related to that of Theorem 2.6. Before proceeding, however, it may be worthwhile to illustrate the method in an alternate proof of Theorem 2.1 (for the case that $\theta(x) \equiv 1$).

Let $\mathcal{G}_1 = \{G_w : 0 \le w \le \zeta^{-1}(v)\}$, where

$$(2.18) \qquad 1 - G_w(x) = \begin{cases} 1, & x \le w \\ e^{-a(x-w)}, & x \ge w \end{cases}$$

and a is determined by

$$(2.19) \qquad \int_0^\infty \zeta(x)\,dG_w(x) = v.$$

Let $\mathcal{G}_2 = \{G_w : \zeta^{-1}(v) \le w\}$ where

$$(2.20) \qquad 1 - G_w(x) = \begin{cases} e^{-bx}, & 0 \le x < w \\ 0, & x \ge w \end{cases}$$

and b is determined by (2.19).

We remark, but do not prove, that a and b are uniquely determined by (2.19).

<u>Alternate Proof of Theorem 2.1, $\theta(x) \equiv 1$.</u> Consider the case that $t < \zeta^{-1}(\nu)$. Let $G_w \in \mathcal{G}_1$, where a_1 satisfies (2.19). We may assume $F \notin \mathcal{G}_1$ and $F(x) > 0$ for $x > t$ since otherwise the inequality is obvious. By log concavity, $1 - F(x)$ must cross $1 - G_t(x)$ from below say at $x_0 > t$. Furthermore, $1 - F(x)$ must be continuous at x_0 and possess a right derivative at x_0. The slope of $1 - G_t(x)$ at x_0 is less than the slope of $1 - F(x)$ at x_0; i.e.

$$f(x_0^+) \leq a_1 e^{-a_1(x_0 - t)} .$$

Since $1 - F(x_0) = 1 - G_t(x_0)$, $q(x_0) \leq a_1$, and by monotonicity of q, $q(t+) \leq a_1$. Since $1 - G_t(t) = 1$, equality is attained in (2.3) by $G_t \in \mathcal{G}_1$. ||

A similar proof of Theorem 2.2 utilizes both $\mathcal{G}_1$ and $\mathcal{G}_2$.

3. <u>Bounds on</u> PF_2 <u>densities and hazard rates</u>. In this section we strengthen the hypothesis that F is IHR by assuming that F has a PF_2 density f. This stronger hypothesis makes possible improvements of the inequalities of Section 2 except in cases where equality is attained by a distribution with PF_2 density (see the remark following Theorem 2.2').

<u>Theorem 3.1</u>. Let f be a PF_2 density such that $f(x) = 0$ for $x < 0$. Let ζ be a function continuous and strictly monotone on $[0,\infty)$ such that $\int_0^\infty \zeta(x) f(x)dx = \nu$ exists finitely. Then

$$(3.1) \qquad q(t) \leq \begin{cases} a_1, & t < \zeta^{-1}(\nu) \\ \infty, & t \geq \zeta^{-1}(\nu), \end{cases}$$

$$(3.2) \qquad f(t) \le \begin{cases} a_1 & t < \zeta^{-1}(v) \\ \infty & t = \zeta^{-1}(v) \\ be^{-bt}/(1 - e^{-bt}), & t > \zeta^{-1}(v), \end{cases}$$

where a_1 is the unique solution to

$$v = \int_0^t \zeta(x) a_1 e^{-a_1(x-t)} dx$$

and b is the unique solution to

$$(3.3) \qquad v = \int_0^t \zeta(x) be^{-bx} dx/(1 - e^{-bt}).$$

All inequalities are sharp.

Proof. (3.1) and (3.2) for $t < \zeta^{-1}(v)$ follow from Theorem 2.1, and sharpness follows from the remark following Theorem 2.2'. In [5, Lemma 5.3], it is proved that for ζ increasing, (3.3) has a unique solution b whenever $t > \zeta^{-1}(v)$; by obvious modifications of the proof given there, we obtain the same result for ζ decreasing. Let

$$g_t(x) = \begin{cases} be^{-bx}/(1 - e^{-bt}), & 0 \le x \le t \\ 0, & x > t \end{cases}$$

and suppose that $f \not\equiv g_t$. Since $\log f(x)$ is concave and $\log g_t(x)$ is linear in $x \epsilon [0,t]$, there are at most two crossings of f by g_t (see [9]). Since f and g_t are densities satisfying $\int_0^\infty \zeta(x) f(x) dx = \int_0^\infty \zeta(x) g_t(x) dx = v$, they cross at least twice. Hence f and g_t cross exactly twice in $[0,t]$; moreover, the second crossing of f by g_t must be from below, and we conclude that $f(t) \le g_t(t)$ as asserted. Of course, equality in (3.2) for $t \ge \zeta^{-1}(v)$ is attained by g_t. ||

Theorem 3.2. Let f be a PF_2 density such that $f(x) = 0$ for $x < 0$. Let $\zeta(x)$ be a function continuous and strictly monotone on $[0,\infty)$ such that $\int_0^\infty \zeta(x)f(x)dx = \nu$ exists finitely. Then

$$(3.4) \qquad q(t) \geq \begin{cases} 0 & t < \zeta^{-1}(\nu) \\ \inf_{m \geq t} g_m(t)/\int_t^m g_m(x)dx, & t \geq \zeta^{-1}(\nu), \end{cases}$$

where

$$g_m(x) = \begin{cases} be^{-bx}/(1 - e^{-bm}), & 0 \leq x \leq m \\ 0, & \text{elsewhere,} \end{cases}$$

and b is uniquely determined by $\int_0^m \zeta(x)g_m(x)dx = \nu$.

Proof. Sharpness of the trivial lower bound for $t < \zeta^{-1}(\nu)$ follows from Theorem 2.2'. Let $x^*(m)$ be the unique point where g_m crosses f from below, and suppose first that $t < x^*(\infty)$. Then there exists $m_0 > t$ such that $f(t) = g_{m_0}(t)$ (the proof of this in case ζ is increasing is given in [5, Proof of Theorem 5.1]; the modifications necessary in case ζ is decreasing are obvious and not extensive). But $f(t) = g_{m_0}(t)$ together with $1 - F(t) \leq \int_t g_{m_0}(x)dx$ (again, see [5, Proof of Theorem 5.1]) yields the desired result.

It remains to consider the case that $t \geq x^*(\infty) \equiv x^*$. Then by an argument identical with the case $t < x^*$ we obtain

$$q(x^*) \geq g_\infty(x^*)/\int_{x^*}^\infty g_\infty(x)dx$$

which together with q increasing yields (3.4) in this case. ||

As noted in Section 2, PF_2 densities have non-trivial lower bounds at "moment points". In particular, we obtain lower bounds on $f(\mu_r^{1/r})$ for $r \geq 1$. To do this, we use the following

<u>Lemma 3.3</u>. If $\int \varphi(x)f_1(x)dx = \int \varphi(x)f_2(x)dx < \infty$, and if the support of f_1 is contained in the support of f_2, then

(3.5) $\qquad \int \varphi(x)f_1(x)\log[f_1(x)/f_2(x)] \geq 0.$

<u>Proof</u>. $\int \varphi(x)f_1(x)\log[f_1(x)/f_2(x)]dx = - \int \varphi(x)f_1(x)\log[f_2(x)/f_1(x)]dx \geq$ $\geq \int \varphi(x)f_1(x)[1 - f_2(x)/f_1(x)]dx = \int \varphi(x)f_1(x)dx - \int \varphi(x)f_2(x)dx = 0$. The inequality follows directly from $\log z \leq z - 1$, $z > 0$. ||

<u>Remark</u>. With $\varphi(x) \equiv 1$, this is the well-known "information inequality".

<u>Theorem 3.4</u>. Let φ be a non-negative function and λ be a number such that

$$0 < \int_0^\infty \varphi(x)f(x)dx = \int_0^\infty \varphi(x)\lambda e^{-\lambda x}dx < \infty.$$

If f is PF_2 and $f(x) = 0$, $x < 0$, then

(3.6) $\qquad f(a) > \lambda e^{-\lambda a}$

where $a = (\int x\varphi(x)f(x)dx)/(\int \varphi(x)f(x)dx)$.

<u>Remark</u>. λ satisfying $\int_0^\infty \varphi(x)f(x)dx = \int_0^\infty \varphi(x)\lambda e^{-\lambda x}dx$ does not necessarily exist in general. However, if φ is monotone, then such a λ always exists.

<u>Proof</u>. Since f is log concave, $\log f(x)$ lies below its tangent at a, i.e., $(x-a)f'(x)/f(a) + \log f(a) \geq \log f(x)$. If $\varphi(x) \geq 0$,

$$\varphi(x)(x - a)f'(a)/f(a) + \varphi(x)\log f(a) \geq \varphi(x)\log f(x)$$

and upon integrating, we obtain

$$\frac{f'(a)}{f(a)} \int_0^\infty \varphi(x)(x - a)f(x)dx + \log f(a) \int_0^\infty \varphi(x)f(x)dx \geq \int_0^\infty \varphi(x)f(x)\log f(x)dx$$

$$\geq \int_0^\infty \varphi(x)f(x)[\log \lambda - \lambda x]dx = (\log \lambda - a\lambda) \int_0^\infty \varphi(x)f(x)dx.$$

The second inequality follows from Lemma 3.3. By the definition of a, the first term on the left of this inequality is zero, and we have

$$\log f(a) \int_0^\infty \varphi(x)f(x)dx \geq (\log \lambda - a\lambda) \int_0^\infty \varphi(x)f(x)dx. \quad \|$$

Corollary 3.5. Let f be a PF_2 density such that $f(x) = 0$ for $x < 0$, and $\mu_r = \int_0^\infty x^r f(x)dx$. Then if $r \geq 1$,

$$f(\mu_r^{1/r}) \geq [\Gamma(r+1)/\mu_r]^{1/r} e^{-[\Gamma(r+1)]^{1/r}}. \tag{3.7}$$

Proof. If $r = 1$, the result follows from Theorem 3.4 with $\varphi(x) \equiv 1$. If $r > 1$, let $\varphi(x) = x^r + (\mu_{r+1} - \mu_r^{(r+1)/r})/(\mu_r^{1/r} - \mu_1)$. Then since $\mu_s^{1/s}$ is increasing in $s > 0$, it follows that $\varphi(x) > 0$. By straightforward algebra, $a = \mu_r^{1/r}$. Thus $\lambda = [\Gamma(r+1)/\mu_r]^{1/r}$, and (3.7) follows. $\|$

The bound of (3.7) for $r = 1$ was originally communicated to us by Samuel Karlin.

4. Bounds on densities and hazard rates when F is IHR with specified first and second moments. Assume that F is IHR with $\mu_1 = 1$ and μ_2 specified. In [5], the class of extremal distributions for bounding F were determined. This same class of distributions is also important in bounding f and q, and we begin with some definitions.

Let $T_0 = 1 - \sqrt{\mu_2 - 1}$ (since F is IHR, $\mu_1^2 \leq \mu_2 \leq 2\mu_1^2$ so $\mu_2 - 1 \geq 0$), and let $T_1 = - a_0^{-1}\log(1 - a_0)$ where a_0 in $[0,1]$ satisfies

$$(4.1) \qquad \mu_2/2 = \int_0^{T_1} e^{-a_0x}dx = a_0^{-1}[1 + \frac{1 - a_0}{a_0}\log(1 - a_0)].$$

Let $\mathcal{G}_3 = \{G_T, T \geq T_1\}$ where

$$(4.2) \qquad 1 - G_T(x) = \begin{cases} 1, & x < \Delta \\ e^{-a(x-\Delta)}, & \Delta \leq x \leq T, \quad T \geq T_1 \\ 0, & x > T \end{cases}$$

and where a and Δ in $[0,T_0]$ are determined by the moment conditions, i.e.,

$$(4.3) \qquad \int_0^\infty [1 - G_T(x)]dx = \mu_1 = 1,$$

$$(4.4) \qquad \int_0^\infty x[1 - G_T(x)]dx = \mu_2/2.$$

Let $\mathcal{G}_4 = \{G_T : T_0 \leq T \leq T_1\}$, where

$$(4.5) \qquad 1 - G_T(x) = \begin{cases} e^{-a_1x}, & x \leq T \\ e^{-a_1T-a_2(x-T)}, & x \geq T \end{cases} \quad ; T_0 \leq T \leq T_1$$

and $a_1 \leq a_2$ are determined by the moment conditions (4.3), (4.4) as before. It is shown in [5, Lemma 3.4] that a,Δ and a_1,a_2 satisfying (4.3) and (4.4) exist uniquely. It is also shown in [5] that for $t \geq 0$,

$$\inf[1 - G_T(t)] \leq 1 - F(t) \leq \sup[1 - G_T(t)]$$

where the extremums are taken over $\mathcal{G}_3 \cup \mathcal{G}_4$. These bounds have been tabulated for $\mu_1 = 1$ and selected values of $\mu_2(1 < \mu_1 \leq 2)$ [1].

<u>Theorem 4.1</u>. If F is IHR with density f, $F(0) = 0$, $\mu_1 = 1$ and μ_2 is specified, then

$$(4.6) \qquad f(t+) \le q(t+) \le \begin{cases} a_0, & t = 0 \\ a, & 0 < t < T_0 \\ (\mu_2 - 1)^{-1/2}, & t = T_0 \\ \infty, & t \ge T_1, \end{cases}$$

$$(4.7) \qquad q(t+) \le a_2, \qquad T_0 \le t \le T_1,$$

and

$$(4.8) \qquad f(t+) \le a_2 e^{-a_1 t}, \qquad T_0 < t < T_1,$$

where a_0 is defined by (4.1); a is defined by $G_T \in \mathcal{G}_3$ with $\Delta = t$ and some $T \ge T_1$; a_1 and a_2 are defined by $G_T \in \mathcal{G}_4$ with $T = t$. All inequalities are sharp.

Proof. Case 1. $0 < t < T_0$. Either $F(t) = 0$ and $f(t) = 0$, or $1 - F(x)$ crosses $1 - G_T(x)$ from below at, say $t_0 \ge t$, where $1 - G_T(x)$ is given by (4.2) with $\Delta = t$. Therefore $f(t_0) \le g_T(t_0)$ and $1 - F(t_0) = 1 - G_T(t_0)$, so that $q(t_0) \le q_T(t_0) = a$, where g_T (q_T) is the density (hazard rate) of G_T. Since $t < t_0$ and q is increasing, we have that $q(t+) \le a$. Equality in $f(t+) \le a$ is attained by the density of G_T. Letting t decrease to zero, we see that $f(0+) \le \lim_{t \downarrow 0} a = a_0$.

Case 2. $T_0 < t < T_1$. From [5, Theorem 3.3] we know that $1 - F(t) \le 1 - G_t(t)$ where G_t is given by (4.5). This together with the fact that F and G_t must cross at least twice implies that $1 - F(x)$ must cross $1 - G_t(x)$ from below at some $t_0 > t$. Hence $q(t+) \le q(t_0) \le q_t(t_0) = a_2$ where q_t is the hazard rate of G_t; and this is (4.7). From (4.7) and $1 - F(t) \le 1 - G_t(t) = e^{-a_1 t}$, we obtain (4.8). Equality is attained by G_t in both (4.7) and (4.8). Letting t decrease to T_0, we obtain from this, (4.6) with $t = T_0$.

Case 3. $t \geq T_1$. The bound $f(t) \leq q(t) \leq \infty$ cannot be improved as can be seen by considering the extremal distribution $G_T \in \mathcal{G}_3$ where $T = t$. ||

Remark. $e^{-a_1 t}$ for $T_0 \leq t \leq T$, is tabulated in Table III, [1].

Theorem 4.2. If F is IHR, $F(0) = 0$, $\mu_1 = 1$ and μ_2 is specified, then

$$(4.9) \qquad q(t-) \geq \begin{cases} 0, & 0 \leq t \leq T_0 \\ a_1, & T_0 < T < T_1 \\ a_0, & t = T_1 \\ a, & t > T_1 \\ (\mu_2 - 1)^{-1/2}, & T = \infty, \end{cases}$$

where a_0, a and a_1 are defined in Theorem 4.1. The inequality is sharp.

Proof. Case 1. $0 \leq t \leq T_0$. The lower bound is attained by $G_T \in \mathcal{G}_4$ for $\Delta \geq t$.

Case 2. $T_0 < t < T_1$. Consider $G_t \in \mathcal{G}_4$. Either $1 - F(x)$ crosses $1 - G_t(x)$ from above in $[0,t]$, or $1 - F(x) \leq 1 - G_t(x)$ for x in $[0,t]$. Suppose $1 - F$ crosses from above, say at $t_0 < t$. Then $q(t-) \geq q(t_0) \geq q_t(t_0) = a_1$ where q_t is the hazard rate of G_t. Next, suppose $1 - F(x)$ lies entirely below $1 - G_t(x)$ for x in $[0,t]$. Then $q(t-) \geq q(0+) \geq a_1$, which completes the proof of this case.

The cases $t = T_1$ and $t = \infty$ are obtained as limit results from Case 2. ||

Theorem 4.3. If F is IHR with density f, $F(0) = 0$, $\mu_1 = 1$, and μ_2 is specified, then

$$(4.10)\qquad f(t-) \geq \begin{cases} 0, & 0 \leq t \leq T_0 \\ \min[g(t),h(t)], & T_0 < t \leq T_1 \\ 0, & t > T_1, \end{cases}$$

where $g(t) = \inf_{0\leq\Delta\leq T_0} ae^{-a(t-\Delta)}$ and $h(t) = \inf_{T_0\leq T\leq T_1} a_1 e^{-a_1 t}$.

Proof. If $0 \leq t \leq T_0$, the lower bounds are attained by G_{T_0}. If $t > T_1$, the lower bound is attained by G_{T_1}. Suppose now that $T_0 < t \leq T_1$. Let $s(T)$ denote the crossing in (Δ,T) from above of $1 - G_T \in \mathcal{G}_3$ by $1 - F$ if such a crossing exists; otherwise, let $s(T) = T$.

Case 1. $T_0 < t \leq s(T_1)$. Let $G_T \in \mathcal{G}_4$, and let $w(T)$ denote the point at which $1 - F(x)$ crosses $1 - G_T(x)$ from above in $(0,T)$. Then $s(T_1) = w(T_1)$, $\lim_{T\downarrow T_0} w(T) = 0$ and $w(T)$ is continuous in T (see [5, Proof of Theorem 3.1]). Hence there exists T such that $w(T) = t$. Since $1 - G_T(x)$ crosses $1 - F(x)$ from below at t, it follows that $f(t-) \geq a_1 e^{-a_1 t}$.

Case 2. $s(T_1) \leq t \leq s(\infty)$. Let $G_T \in \mathcal{G}_3$; by continuity of $s(T)$, there exists T such that $s(T) = t$. Since $1 - F(x)$ crosses $1 - G_T(x)$ from above at t, $f(t-) \geq ae^{-a(t-\Delta)}$.

Case 3. $s(\infty) \leq t \leq T_1$. If $s(\infty) > T_1$, then of course this case is vacuous. Otherwise, let $v(T)$ be the crossing in (T,∞) from above of $1 - G_T(x)$ by $1 - F(x)$ if such a crossing exists, and let $v(T)$ be the right-hand endpoint M of the support of F if such a crossing does not exist. Then $v(T_0) = s(\infty)$ and $\lim_{T\uparrow T_1} v(T) = M$. By continuity of v, there exists T in $[T_0,T_1]$ such that $1 - F(x)$ crosses $1 - G_T(x)$ from above at t, and the argument is concluded as in the previous cases. $\|$

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